

BINARY REPRESENTATION

Representations of Integers

- In the modern world, we use decimal, or base 10, notation to represent integers.
- We can represent numbers using any base b , where b is a positive integer greater than 1.

Base 10

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 - $9 \cdot 10^2 + 6 \cdot 10^1 + 5 \cdot 10^0$

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 - $9 \cdot 10^2 + 6 \cdot 10^1 + 5 \cdot 10^0$
- $n = a_k b^k + a_{k-1} b^{k-1} + \dots + a_1 b + a_0$

Base b

- **Theorem:** Let b be a positive integer greater than 1. Then if n is a positive integer, it can be expressed uniquely in the form:

$$n = a_k b^k + a_{k-1} b^{k-1} + \dots + a_1 b + a_0$$

where k is a nonnegative integer and a_0, a_1, \dots, a_k are nonnegative integers less than b .

- This representation of n is called the base b expansion of n and can be denoted by $(a_k a_{k-1} \dots a_1 a_0)_b$.
- We usually omit the subscript 10 for base 10 expansions.

Binary Expansions

- Computers represent integers and do arithmetic with binary (base 2) expansions of integers. In these expansions, the only digits used are 0 and 1.

Binary Expansions

- **Example:** What is the decimal expansion of the integer that has $(11011)_2$ as its binary expansion?

- **Solution:**

$$(11011)_2$$

$$= 1 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0$$

$$= 16 + 8 + 0 + 2 + 1$$

$$= 27.$$

Binary Expansions

- **Example:** What is the decimal expansion of the integer that has $(1\ 0101\ 1111)_2$ as its binary expansion?

- **Solution:**

$$(1\ 0101\ 1111)_2$$

$$= 1 \cdot 2^8 + 0 \cdot 2^7 + 1 \cdot 2^6 + 0 \cdot 2^5 + 1 \cdot 2^4 + 1 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0$$

$$= 256 + 0 + 64 + 0 + 16 + 8 + 4 + 2 + 1$$

$$= 351.$$

Base Conversion

To construct the base b expansion of an integer n :

- Divide n by b to obtain a quotient and remainder.

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$$q_0 = bq_1 + a_1 \quad 0 \leq a_1 < b$$

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- Next, divide q_0 by b .
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- The remainder, a_1 , is the second digit from the right in the base b expansion of n .
- Continue by successively dividing the quotients by b , obtaining the additional base b digits as the remainder. The process terminates when the quotient is 0.

Base Conversion

- **Example:** Find the binary expansion of $(19)_{10}$
- **Solution:** Successively dividing by 2 gives:
 - $19 = 2 * 9 + 1$
 - $9 = 2 * 4 + 1$
 - $4 = 2 * 2 + 0$
 - $2 = 2 * 1 + 0$
 - $1 = 2 * 0 + 1$

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 - $4 = 2 * 2 + 0$
 - $2 = 2 * 1 + 0$
 - $1 = 2 * 0 + 1$
- The remainders are the digits: read from bottom to top to become the binary number from left to right
- $(10011)_2$.

Base Conversion

- **Example:** Find the binary expansion of $(12345)_{10}$
- **Solution:** Successively dividing by 2 gives:
 - $12345 = 2 * 6172 + 1$
 - $6172 = 2 * 3086 + 0$
 - $3086 = 2 * 1543 + 0$
 - $1543 = 2 * 771 + 1$
 - $771 = 2 * 385 + 1$
 - $385 = 2 * 192 + 1$
 - $192 = 2 * 96 + 0$
 - $96 = 2 * 48 + 0$
 - $48 = 2 * 24 + 0$
 - $24 = 2 * 12 + 0$
 - $12 = 2 * 6 + 0$
 - $6 = 2 * 3 + 0$
 - $3 = 2 * 1 + 1$
 - $1 = 2 * 0 + 1$
- $(12345)_{10} = (11\ 0000\ 0011\ 1001)_2$

List of Binary Numbers

Decimal	Binary
0	0000
1	0001
2	0010
3	0011
4	0100
5	0101
6	0110
7	0111

- These are 4-bit unsigned binary numbers.

Binary Addition

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Binary Subtraction

- How do we subtract numbers?
- $7 - 6 = 7 + (-6) = 1$
- We need a way to represent negative numbers

Signed Binary Numbers

- To represent a negative number, we use a sign bit.
- The sign bit is the most significant bit (MSB)
 - 1 represents a negative number
 - 0 represents a positive number

Signed Representations

- Sign-Magnitude: sign bit + absolute value
 - Positive 7 0111
 - Negative 7 1111
- 1's Complement: bitwise inverse
 - Positive 7 0111
 - Negative 7 1000
- Disadvantages
 - Two representations for 0: +0, -0
 - Arithmetic is more complex
- Advantages
 - The magnitude (absolute value) of the smallest number is the same as the largest number

2's Complement Representation

- To represent a **negative** value: do a bitwise inverse, then add 1
- To determine -7:
 - 7 0111
 - Bitwise inverse 1000
 - Add 1 1001
- 2's Complement is the standard representation for negative numbers.

Binary Numbers

List of Binary Numbers in 2's Complement Representation

Decimal	Binary	Decimal	Binary
0	0000	-1	1111
1	0001	-2	1110
2	0010	-3	1101
3	0011	-4	1100
4	0100	-5	1011
5	0101	-6	1010
6	0110	-7	1001
7	0111	-8	1000

4 bits gives us the following range:

Biggest 4-bit number: 7

Smallest 4-bit number: -8

Binary Subtraction

- 2's complement binary numbers are subtracted by adding their negative equivalent.
- $7 - 6 = 7 + (-6) = 1$

	0	1	1	1
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1

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- The extra 1, called the carry-out, is ignored

Overflow

- Overflow occurs when the result of an arithmetic operation is too large or too small to represent.
 - In our 4-bit examples, that would occur if the result is less than -8 or greater than 7.

Overflow

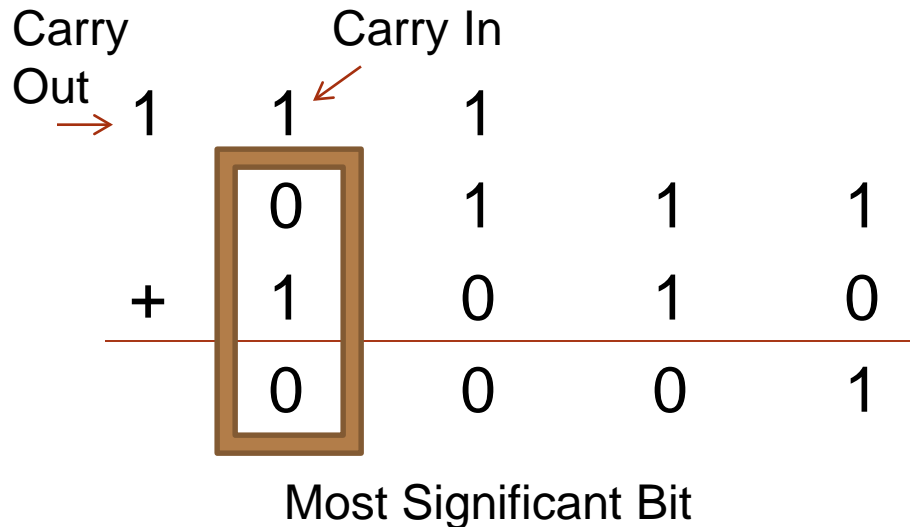
- Overflow occurs when adding whenever:
 - 2 positive numbers are added and the result is negative
 - 2 negative numbers are added and the result is positive

Overflow

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Since the Carry in of the MSB is equal to the carry out, no overflow has occurred. The carry out is discarded and “0001” is the correct result.

Overflow

- An easy way to spot overflow is to compare the “Carry in” of the MSB to the “Carry out”. If they are different, then overflow has occurred.
- $7 + 3 = 10$

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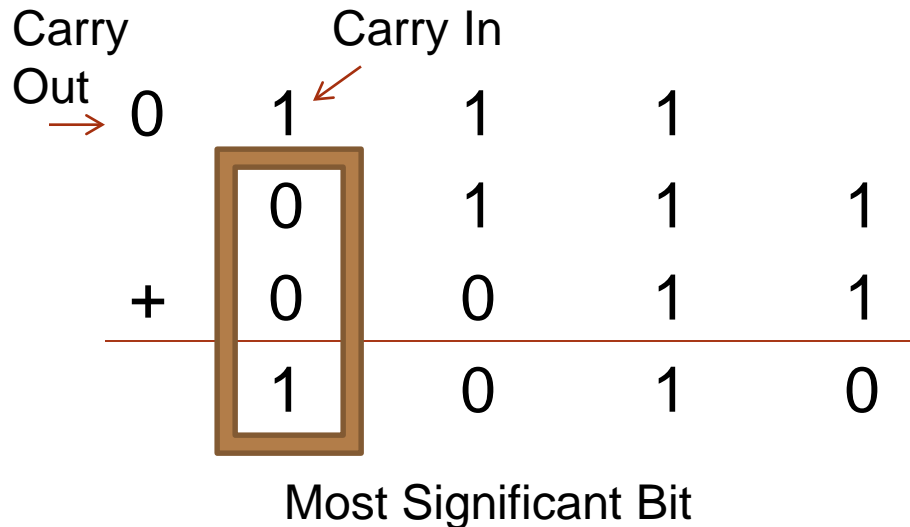
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Since the Carry in of the MSB is not equal to the carry out, an overflow has occurred.

The result calculated (-6) is clearly incorrect.

Overflow

- An easy way to spot overflow is to compare the “Carry in” of the MSB to the “Carry out”. If they are different, then overflow has occurred.
- $-4 - 5 = -9$

	1	1	0	0
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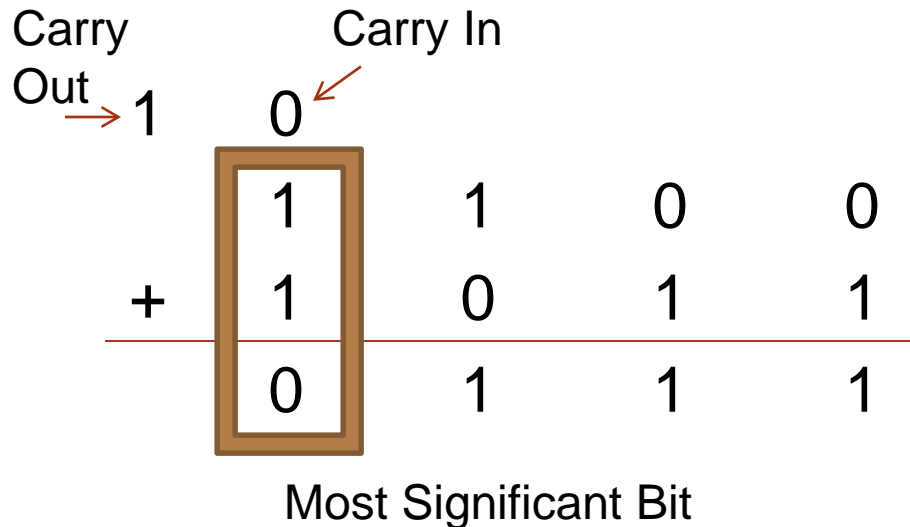
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The result calculated (7) is clearly incorrect.

Overflow

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- $3 + (-5) = -2$

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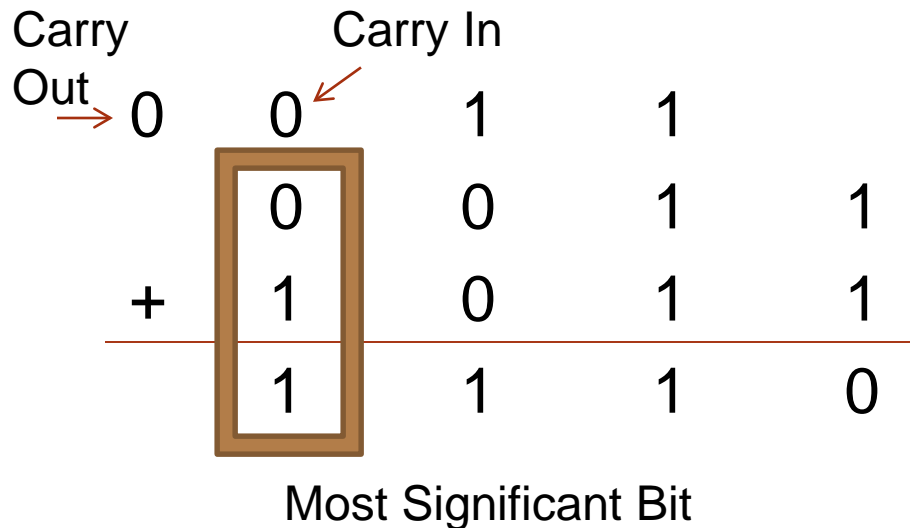
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Since the Carry into of the MSB is equal to the carry out, no overflow has occurred. The extra zero is discarded and “1110” is the correct result.