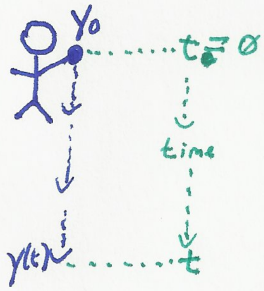
# Differential Equations

## I. Simple ODE's by Example

- If we want to treat many real world properties like time properly, we must note that many of these properties are continuous
- For example, we may Not want to model time by discrete steps, but instead construct mathematical models that incorporate the continuity of time (or other properties)
- When we do this, it's often most ~~convenient~~ natural to construct our models as functions of change

### Example: Dropping a Ball




If the ball is let go at some initial height, $Y_0$, how will it fall? (Ignoring friction and obstructions)

$$\boxed{\text{The ball will accelerate at a constant rate downward } (-g).}$$

So we might model the change in $y$ as $-gt$.
In differential equations, we might have written this:

$$y' = -gt,$$

which is equivalent to: $\dfrac{dy}{dt} = -gt$

This model makes sense as a function of change.

But can we find the explicit function $y(t)$ that gives us the precise position of the ball at any time?

+ Separation of variables: let's collect our terms, including to differentials on either side of the equation:

$$dy = -gt\ dt$$

+ Then integrate both sides:

$$\int dy = \int -gt\ dt$$

$$y = -\tfrac{1}{2}gt^2 + C$$

what is $C$? Consider the solution at $t = \emptyset$

$$Y_0 = Y(\emptyset) = -\tfrac{1}{2}g(\emptyset)^2 + C = C$$

So: $\boxed{y(t) = -\tfrac{1}{2}gt^2 + Y_0}$ solution of the D.E.!

# Differential Equations



**Example: Population Growth**: On the first day, we considered a discrete-time model such as $\underbrace{n(k+1)}_{\text{Population at time } k+1} = \underbrace{n(k)}_{\text{Population at time } k} \cdot \underbrace{r}_{\text{Growth Rate}}$

- But suppose you want to model growth that is cts. over time?
  - + Note that the growth rate, $r$, is a ratio of the change in population size over the population size
  - + If we expect $r$ to be constant, then:

$$r = \frac{\Delta}{n}$$

$\Delta$ ← change in population size

$n$ ← population size (which is a function of time)

Note that:

$$\Delta = \frac{dn}{dt}, \text{ by definition}$$

So we have:

$$r = \frac{\left(\frac{dn}{dt}\right)}{n}$$

Separate our variables:

$$r = \frac{1}{n}\left(\frac{dn}{dt}\right)$$

$$r\,dt = \frac{1}{n}\,dn$$

Integrate:

$$\int r\,dt = \int \frac{1}{n}\,dn$$

$$rt + c_1 = \ln n + c_2$$

$$\ln n = rt + c$$

Since $c_1$ & $c_2$ are constants I just combined them $c = c_1 + c_2$

$$e^{\ln n} = e^{rt} \cdot e^{c}$$

$$n = e^{rt} \cdot C$$

Since $e^c$ is also a constant, I just combined them: $C = e^c$

What is $C$? Consider $t = \emptyset$, then

$$n_0 = n(\emptyset) = e^{r \cdot \emptyset} \cdot C = 1 \cdot C = C$$

So $C$ is the initial population size

**Our solution:**

$$n(t) = n_0 \cdot e^{rt}$$

← tells us the population size at time $t$

- Notice that a solution to a D.E. is typically not a point or a curve, but a family of curves.
- But we mean "solution" in the same way we always have:
  - + Given some (set of) assertion(s) of fact, under what conditions do these facts hold true?
- In Differential Equations, these "assertions of fact" are expressed as functions of change.
- And solutions tend to be things that give us properties of the system at certain positions (eg, where is the ball at time $t$).

# II. Terms & Types of D.E.'s

Defn: A differential equation is an equation involving derivatives of a function or set of functions.

- There are two main types:

Ordinary Differential Equations (ODE's) - An equation that depends only on one independent variable:

independent variable t

$$\frac{dy}{dt} = Ky(t)$$

dependent variable y

or:

dependent variable y

$$\frac{dy}{dt} = \frac{2(y(t)-1)}{x(t)}$$

independent variable t

dependent variable x

Partial Differential Equations (PDEs) - An equation in which the dependent variable depends on two or more independent variables:

dependent variable

independent variables

$$\frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2} = \emptyset$$

- We can also classify them by their order
  - The order of a differential equation is the order of its highest derivative:

second derivative

$$m\frac{d^2x}{dt^2} = -Kx$$

] A second order ODE

- Or by linearity:

  + A linear D.E. (first order) is one that can be written in the form $\frac{dy}{dt} + g(t)y = r(t)$, where $g(t)$ and $r(t)$ are arbitrary functions of $t$:

$$\frac{dy}{dt} = t^2 y + \cos(t)$$

Alt. Defn (but equiv.): A D.E. is linear if the highest power of any dependent variable or its derivatives is 1. So $\frac{d^2y}{dt} + y = t$ is linear but $\left(\frac{dy}{dt}\right)^2 + y = t$ is not

  + We can do this for any order:

$$\frac{d^2y}{dt^2} + h(t)\frac{dy}{dt} + g(t)y = r(t)$$

  + A non-linear D.E. is one whose right hand side is not a linear function of the dependent variable(s):

$$\frac{dx}{dt} = K\left(1 - \frac{x}{n}\right)x$$

non-linear

- Or by Homogeneity:

  + A Homogeneous linear D.E. is one whose right hand side is $\emptyset$ that is, in linear form $r(t) = \emptyset$:

$$\frac{dy}{dt} + Ky = \emptyset$$

  + A Non-Homogeneous (or sometimes "inhomogeneous") linear D.E. cannot be put in a linear form such that the RHS is $\emptyset$:

$$\frac{dy}{dt} + 2y(t) = \sin(2t)$$

- Testing ourselves (cover up the RHS to see if you can figure out what these are w/o looking)

| | |
|---|---|
| $\frac{du}{dx} = cu + x^2$ | Inhomogeneous first order linear ODE (notice that $x$ is the indep var.) |
| $\frac{d^2u}{dx^2} - x\frac{du}{dx} + u = \emptyset$ | Homogeneous second order linear ODE |
| $\frac{d^2u}{dx^2} + \omega^2 u = \emptyset$ | Homogeneous second order liner ODE |
| $\frac{du}{dx} = u^2 + 1$ | Inhomogeneous first order non-linear ODE |
| $L\frac{d^2u}{dx^2} + g\sin u = \emptyset$ | Homogeneous second order non-linear ODE |
| $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \emptyset$ | Homogeneous second order linear PDE |
| $\frac{\partial u}{\partial t} = 6u \cdot \frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x^3}$ | Third-order Non-linear PDE |

- Notation: Sometimes derivatives are written:

| | | | |
|---|---|---|---|
| $\frac{dy}{dt}$ | $y'$ | $\dot{y}$ | First derivative of $y$ w.r.t. $t$ |
| $\frac{d^2y}{dt^2}$ | $y''$ | $\ddot{y}$ | Second derivative of $y$ wrt $t$ |
| $\frac{\partial u}{\partial x}$ | $u_x$ | | Partial derivative of $u$ wrt $x$ |
| $\frac{\partial^2 u}{\partial x \partial y}$ | $u_{xy}$ | | Partial second derivative of $u$ wrt $x$ then $y$. |

Also: the unknown/dependent variable somes written $y$ and sometimes written $y(t)$.

## III. Views of D.E.s, Solutions & Problems

- In terms of analyzing/solving D.E.'s, there are three "views"

  1. Analytical – where our goal is to find an equation (or equations) for our unknown/dependent variable(s) from the D.E.

  2. Geometric – where our goal is to find a graphical solution to the D.E. by sketching a "directional field" and one or more "integral curves"

  3. Numerical – where our goal is to find approximate solutions by discritizing the system in some way

In 1st order ODE speak:

$y' = f(x, y) \iff$ directional field

$y_1(x) \iff$ Integral curve

- Today we'll talk about 1 and 2; we'll leave 3 for next week.

- When we solve differential equations, we're often interested in the answer to one of two broad classes of problems:

  1. Initial value problem (IVP) – Find a particular solution from a general set of solutions given ~~some constants typically~~ the initial conditions. Eg., Dropping the ball!

     General solution: $y(t) = -gt^2 + y_0$

     Particular solution: $y(t) = -gt^2 + 10$

  2. Boundary value problem (BVP) – Find a particular solution from a general set of solutions given some set of additional constraints

     Eg.: $y''(x) + y(x) = 0$, subject to $y(0) = 0$ and $y(\pi/2) = 2$

     General Solution: $y(x) = A\sin(x) + B\cos(x)$

     Particular Solution: $y(x) = 2\sin(x)$

# IV. Analytic Solution by example

- We've already seen one tool: separate variables and integrate
- One can also use subtitution tricks to turn second-order ODEs into a system of first-order ODE's

$$x'' + bx' + \sin x = \emptyset$$

Let $y_1 = x$

$y_2 = x'$

$$y_2' + b y_2 + \sin y_1 = \emptyset$$

$$y_2' + b y_1' + \sin y_1 = \emptyset$$

- But linear homogeneous ODEs are "special"...

If $y_1, \ldots, y_n$ are linear and independent solutions to such an ODE:

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_1 \frac{dy}{dt} + a_0 y = \emptyset$$

Then the general solution is:

$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$, where $c_1, \ldots, c_n$ are arbitrary constants

$\rightarrow$ we have reduced this to finding a collection of $n$ indep. particular soln.

Let's consider: $y'' + ay' + by = \emptyset$

We'll "guess" and substitute $e^{mt}$ for each $y_i$ ... note that if $y = e^{mt}$

then $y' = me^{mt}$

and $y'' = m^2 e^{mt}$

So then we have

$$m^2 e^{mt} + ame^{mt} + be^{mt} = \emptyset$$

$$e^{mt}(m^2 + am + b) = \emptyset$$

$e^{mt} \neq \emptyset$, $\therefore$ $y = e^{mt}$ is a solution if $m$ is a solution to

$\underline{m^2 + am + b}$, Characteristic Equation

For example, suppose $a = 0$ and $b = -4$:

$$\left.\begin{aligned} m^2 + 0 \cdot m - 4 &= 0 \\ m^2 - 4 &= 0 \end{aligned}\right\}$$ So $y_1 = e^{m_1 t}$ and $y_2 = e^{m_2 t}$ if $m = \pm 2$

we need lin. indep soln., so we choose $m_1 = +2$, $m_2 = -2$

Solutions are: $y_1 = e^{2t}$, $y_2 = e^{-2t}$

General Solution:

$$y = c_1 e^{2t} + c_2 e^{-2t}$$

More generally for $m^2 + am + b = 0$

i) if $m_1 \neq m_2$ and they are Real (not complex)

then $y = c_1 e^{m_1 t} + c_2 e^{m_2 t}$

ii) if $m_1 = m_2$ and it is Real

then $y = c_1 e^{mt} + c_2 t e^{mt} = (c_1 + c_2 t) e^{mt}$

iii) if $m_1$ and $m_2$ are complex conjugates, $\alpha + Bi$ and $\alpha - Bi$

$$y = c_1 e^{\alpha t} \cos(Bt) + c_2 e^{\alpha t} \sin(Bt)$$

Damped Harmonic Oscillator

$$F = \underbrace{-kx}_{\text{change in position by factor } k} \underbrace{- c\frac{dx}{dt}}_{\text{viscous damping}} = m\frac{d^2x}{dt^2}$$ Newton: $F = ma$

Rewritten: $x'' + 2\zeta_c \omega_0 x' + \omega_0^2 x = 0$, where $\omega_0 = \sqrt{\frac{k}{m}}$, $\zeta_c = \frac{c}{2m\omega_0}$

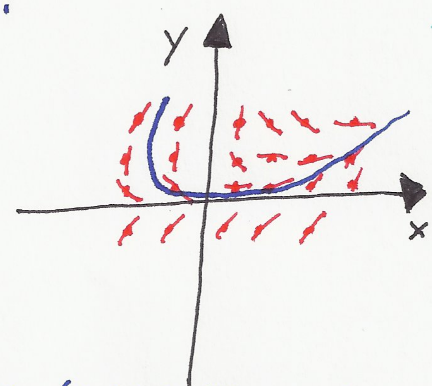
And can be solved by the above method, dep. on constants

- If $\zeta_c > 1$, the system is overdamped and will find equilibrium w/o oscillation (slowly)
- If $\zeta_c = 1$, the system is critically damped and will find eq. w/o oscillation as quickly as possible
- If $\zeta_c < 1$, the system is underdamped and the system oscillates

$\omega_1 = \omega_0\sqrt{1 - \zeta_c^2}$ gives the angular frequency

## V. Geometric Solution by Example

- Since first order ODE's can be written as $y' = f(x,y)$, we can sketch a directional field of a 2 variable system by simply placing points on a ~~plane~~ plane and drawing short line segments indicating the tangent value:



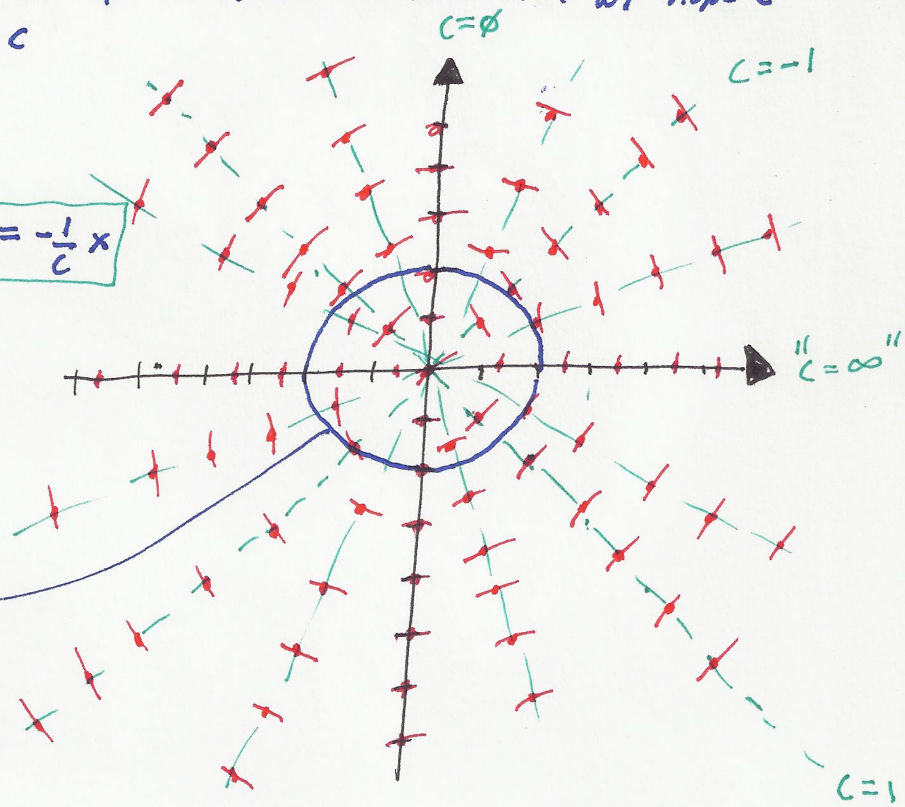

A computer might just go through an evenly spaced set of $(x,y)$ points and compute $f(x,y)$ to use as the tangent

- We can the sketch a curve through the field such that the curve's tangent is the same as the tangent of the point in the directional field at every pt through which it goes (integral curve)

- We can draw the field using isoclines:
  - + set $y'$ to a particular value (c), then plot $f(x,y) = c$
  - + Draw little line segments every so often along the curve w/ slope c
  - + Repeat with a different c

- For example: $y' = -\frac{x}{y}$

  Isoclines: $-\frac{x}{y} = c \rightsquigarrow \boxed{y = -\frac{1}{c}x}$




The integral curves must be perpindicular to all rays going throug the origin ... must be circles

So the solution must be something like
$x^2 + y^2 = C_1^2$

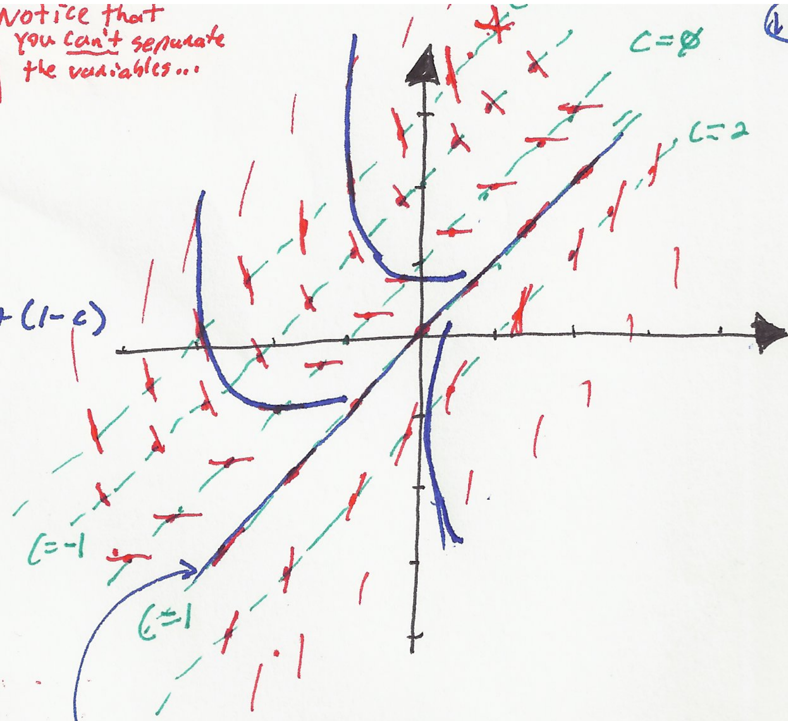
• Another example: $y' = 1 + x - y$

Notice that you can't separate the variables...

For the isoclines: $1 + x - y = c$

or: $-y = c - 1 - x$

$y = 1 + x - c = x + (1 - c)$




Both an isocline and an integral curve so $y = x$ is one solution

But:
What's happening b/w $C=0$ and $C=2$?

+ No escape is possible from solution once it's in that region...

+ Actually, they asymptotically approach $y = x$, but never get there

+ So in the general solution, curves asymptotically approach $y = x$ as $x \to \infty$

**Note that:**

① Two integral curves cross at an angle (X)
Because if one were in the position of the intersection, the next step is the same (can't have two slopes at same point)

② Two integral curves cannot be tangent
(Cannot touch)

## VI Lanchester Equations (Dev. in 1914 by a British Eng.)

X: Number of troops of one side

Y: Number of troops of another side

a: effectiveness of X-side

b: effectiveness of Y-side

t: time

$$\frac{dx}{dt} = -ay$$

x troop populations ~~change~~ decrease by y's effectiveness times their #

$$\frac{dy}{dt} = -bx$$

Y troop populations change by x's effectiveness times their #

~~If you assume the two forces begin equally matched... $x(0) = y(0)$~~
~~Solution is:~~
~~$ay^2 = bx^2$ "fighting strength" of x~~

In general, the solution is:

$$b(x_0^2 - x^2) = a(y_0^2 - y^2)$$

"Square Law" ~ emphasizes advantage of concentration

$\sqrt{ab}$: msrs battle "intensity"

$\sqrt{\frac{a}{b}}$: msrs relative "effectiveness"

Diff. Eq.



- Questions it can answer:
  - + who will win
  - + what force ratio is required for victory
  - + How many survivors the winner will have
  - + How long the battle will last
  - + Etc.

- We can rewrite the soln in terms of x

$$x(t) = \frac{1}{2}\left(\left(x_0 - \sqrt{\frac{a}{b}}\,y_0\right)e^{\sqrt{ab}\,t} + \left(x_0 + \sqrt{\frac{a}{b}}\,y_0\right)e^{-\sqrt{ab}\,t}\right)$$

Gives the population size of X troops over time...

y(t) is analogous

- X wins iff $\frac{x_0}{y_0} > \sqrt{\frac{a}{b}}$

- If they fight to the finish and x wins, $x_f$ is the # troops at end:

$$x_f = \sqrt{\left(x_0^2 - \frac{a}{b}y_0^2\right)}$$

- And it takes:

$$t_f = \frac{1}{2\sqrt{ab}} \ln\left(\frac{1 + \frac{y_0}{x_0}\sqrt{\frac{a}{b}}}{1 - \frac{y_0}{x_0}\sqrt{\frac{a}{b}}}\right)$$

- Similar expressions can be derived for alt. term conditions (eg., x terminates fight if fall below some # & v/v)